An investigation into the practical uses of spherical non-Euclidean geometry in games

Is spherical non-Euclidean geometry in the Unity game engine practical?

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# Introduction

## Justification

Euclidean geometry is defined by a collection of rules as set out in “The Elements” (Euclid, 300 BCE) and has formed the basis of geometry in all modern society, see *figure 1 ‘zero curvature’*. As a result of Euclidean geometry being a key aspect of society, it has an influence on video games as well, “Since the conception of the first video game, its visual representation has always been done mainly in the fashion of the Euclidean geometry.” (Guimaraes, Mello & Velho, 2015).

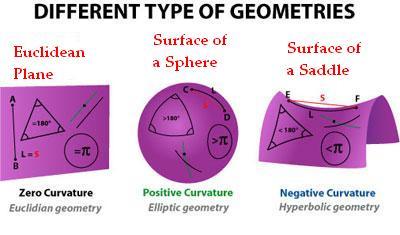


Figure 1: The Difference Between Euclidean and Non-Euclidean Geometry (Kurniadi, 2011).

Wolf (2018) states that “video games can easily allow for non-Euclidean configurations”. However, Taylor (2003) also says “In keeping with these cultural prejudices, descriptions of the optical space of video games presume an uncomplicated optical scheme, founded on traditions of linear perspective.” This suggests that, despite non-Euclidean geometry being, theoretically, easy to implement, a complicated form of non-Euclidean geometry needs to be done to a high quality to be successful. Currently, to make games intuitive and familiar to players the industry heavily relies on the conventions set out in film and television (Wolf, 2004) this means that game designers may have neglected the possibilities that abstract design methods provide. This has the potential to be one of the key aspects of why there is a lack of games that utilise non-Euclidean geometry.

The output design will involve, but not be limited to, an environment containing a procedurally generated maze that is created in a spherically non-Euclidean world. In this a basic character controller will be present, allowing for the movement within the maze. However, there is no intent to have any gameplay elements present, such as UI or a formal gameplay loop. However, this is anticipated to change as development progresses if it would benefit the demonstration of a spherically non-Euclidean geometry. Due to time considerations on the other hand, the main focus in the output Design will be that of the spherically non-Euclidean space meaning these further elements are not likely to be included.

## Aims

This paper aims to investigate the uses, and potential uses, of spherical non-Euclidean geometry, research why non-Euclidean geometry is typically not used in video games, and try to implement a viable method for the implementation of a spherically non-Euclid geometry in the Unity game engine (Unity Technologies, 2020).

## Objectives

* Conduct a review of academic literature regarding non-Euclidean geometry.
* Research and evaluate the uses, and potential uses, of spherical non-Euclidean geometry in the industry.
* Determine a system for simulating spherical non-Euclidean geometry.
* Create a program in the Unity game engine (Unity Technologies) using said system.

# Literature Review

## Overview

This literature review will investigate how computers can simulate Non-Euclidean Geometry. With this, the methods of simulating Non-Euclidean Geometry will be compared whilst also analysing their practical uses within video games. This will then be used to justify the use of Spherical Non-Euclidean Geometry as a viable form of Geometry in games.

## Euclidean Geometry

Euclidean Geometry is the typical form of geometry most people are used to. In this form of geometry, the world around us is mapped onto a plane with no curvature. With this “Each ordered pair (p1, p2) of real numbers determines exactly one point P of the plane. The point determined by (0, 0) is called the origin” (Ryan, 1986). This is the basic coordinate system that we interpret our flat geometry in, with a third real number being used to allow us to interpret in a third dimension. This allows for people to interpret the transform of something with relative ease due to the 2 / 3 axes being easily interpreted. This is especially the case when computers are involved, as the use of transformation matrices allow the storing of data that not only stores the position of an object, but also its rotation and scale.

Defined by Euclid “A line which lies evenly between its extreme points is a straight or right line” (Casey, 2007). This is simple to visualise in a Euclidean setting. Two points have an obvious shortest distance. This makes this definition of a line extremely useful when rendering geometry in a virtual environment, as any two points are linearly connected.

Another, and possibly one of the most important, aspect of Euclidean geometry is that “If a straight line falling on two straight lines make the alternate angles equal to one another, the lines will be parallel to one another” (Wolfe, 2012). This is a key aspect of Euclidean geometry as it forms the basis of many concepts in mathematics. One of these is that “The sum of the angles of a triangle equals two right angles” (Ravindran, 2007), see *figure 2.* Due to this being a key, base principal of geometry the requirement for a consistent idea such as all triangles having an interior angle of 180° is greatly important, and when considered in a flat Euclid plane can help solve other geometric calculations.

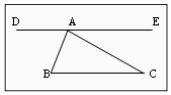


Figure 2: Display to show Euclid's fifth postulates relation to triangles (Ravindran, 2007).

## Non-Euclidean Geometry

### What is Non-Euclidean Geometry

Non-Euclidean geometry exists in two main forms Hyperbolic and Spherical (Manning, 1963). The primary difference between not only these two geometries but also Euclidean geometry is that of curvature. Euclidean geometry, or flat geometry, has a zero curvature nature people are accustomed to. However, it is in non-Euclidean geometry this begins to change. Hyperbolic geometry is a geometry with a negative curvature, whilst spherical, or elliptic, is a geometry with a positive curvature.

### Hyperbolic Non-Euclidean Geometry

Hyperbolic non-Euclidean geometry is one of the major forms of non-Euclidean Geometry found within the games industry. This is due to the fact that it can have an interesting effect on how games are played. This is especially the case when we look at 2D tiled games. “That would not be very interesting in a Euclidean tiling, but hyperbolic geometry makes it fun—not only because it feels very different, but also because it is possible to use the hyperbolic properties tactically, to deal with large groups of enemies” (Kopczynski et al, 2017). This can be credited to the larger areas the game plays out in, due to the curvature found within Hyperbolic geometry.

Hyperbolic geometry has other aspects that deviate from Euclidean Geometry, one of the first is that triangles will have an area less than 180°, “Triangles in hyperbolic space have angle sum less than π; in fact, the area of a triangle with angles α, β, and γ is π − α − β – γ” (James et al, 1997). However, as stated by Euclid’s 5th postulate triangles should have a total interior angle of 180°, this confliction is one of the reasons Hyperbolic geometry is classified as non-Euclidean.

As a result of the triangles in Hyperbolic geometry having less than 180° as their total interior angle, it is evident that Euclid’s 5th postulate does not always apply in Hyperbolic space. However, in Hyperbolic geometry lines can still be parallel. This is due to the different ways in which lines are considered due to the curvature in Hyperbolic geometry. Lines may not intersect, allowing for the opportunity for them to be parallel. This is supported by the fact that a parallel ruler can be utilised in a hyperbolic space. “A parallel-ruler is, as in Euclidean geometry, an instrument for drawing a line through a given point and parallel to a given line. We shall also, as in Euclidean geometry, use the parallel-ruler as an ordinary ruler. THEOREM 1. Any construction in hyperbolic geometry that can be performed by means of a ruler and any of the three compasses, can be performed by means of a parallel-ruler.” (Handest, 1956)

### Spherical Non-Euclidean Geometry

One way that spherical geometry deviates from Euclidean geometry is that lines in a spherical geometry cannot be parallel. This is due to the fact that any straight lines (using Euclid’s definition of a straight line) made in Spherical geometry will always intersect. This is attributed to them creating a greater circle. A greater circle is called this as it exactly bisects the sphere it is on. If we take a globe with lines marked on the longitude and latitude it is apparent that all the latitude lines intersect at the poles, and also forms a circle all around the globe bisecting it. However, if we look at the longitude lines it looks like these lines are also straight but do not intersect, nor do they bisect the sphere. This issue with this is that the lines are not straight and are instead curved. As a result of this, they cannot be parallel (Whittlesey, 2019). The only longitude line that is straight exists on the equator, however, this cannot be parallel for a line to be straight it would have to intersect this line in some way. This means that Euclid’s parallel (5th) postulate cannot apply in a Spherical geometry, making it non-Euclidean.

It is not just Euclidean geometry that can have practical applications in the real world. In fact, the opposite of this is the case as “we still cannot decide whether the real world is approximately Euclidean or approximately non-Euclidean” (Coxeter, 2019). A simple example of this is simply the curvature of the Earth, “Here is a very simple example: Consider the geographical triangle Amsterdam (latitude 528 210 north, longitude 48 540 east), Boston (428 200 north, 718 50 west), and Cape Town (358 560 south, 188 220 east). This triangle has an area of 38 700 000 km2 . 70 J J Koenderink, A J van Doorn, J S Lappin From Amsterdam, Boston is 5568 km at 208 300 north of west, Cape Town is 9914 km at 108 520 east of south, etc. The angle sum of the triangle is 2348 28.30 : This differs appreciably from the Euclidean 1808; thus the Earth is significantly curved.” (Koenderink et al, 1999). This presents a use of Spherical geometry in the real world, as the mathematics for calculating the area of the surface of the earth can only be truthfully representative in a Spherical geometry.

### Rendering Non-Euclidean Geometry

A picture containing text, athletic game, sport, basketball

Description automatically generatedOne of the primary methods of rendering a Hyperbolic geometry on a flat, Euclid, plane is by utilising the Poincaré projection, or Poincaré’s Disk. In this method, a line on the Hyperbolic plane is seen from point 0 on the flat plane. From this, the points on the Hyperbolic plane are then mapped to a flat plane forming a curve from what would have been a straight (Hyperbolic) line, see *figure 3*. However, this form of projection (when done in 3D) can lead to depth issues when resolving the rendered geometry in the fragment shader. This is due to the fact that the curved line projected onto the Euclid plane will be interpreted linearly by the fragment shader. However, due to the curve these lines what to be interpreted non-linearly. As a result of this, the depth of rendered models may appear incorrectly due to the depth of some rendered vertexes being interpolated be behind or in front of where they should be.

Figure 4: Poincaré Projection (Kallosh & Linde 2015).

However, to fix this the Beltrami Klein disk Model can be used. To prevent this depth issue as it is designed to preserve the distance between points. In this method, a similar system is used but instead of mapping the points as a curve, the points are used to construct a plane that is then used to create a chord in a flat plane (Francis, 2002). These points have then retained the scale of their distances and formed a straight line instead of a curved one. This retaining of distance prevents the depth issues mentioned previously.

When rendering Spherical geometry, it is important to utilise the correct form of projection in order to display the geometry in a virtual setting. One such form of projection is that of Gnomonic projection. In this form of projection, the centre of the sphere our geometry exists on is taken as the origin. From here a point on the flat plane we are projecting to is taken as the resulting point. A line is then created between the origin and the resulting point. The position this line then intersects on the sphere is the point in the geometry that will be rendered at the resulting point on the flat plane (Coxeter, 1989), see *figure 4 Gnomonic Projection*. There is, however, an issue with this form of projection, it cannot render the equator of the sphere. This is the case due to the fact that the resulting line from the flat plane to the origin will infinitely move towards the equator of the sphere as the resulting point on the plane infinitely becomes further away from the sphere. If both hemispheres of the sphere are rendered a line will appear on the flat plane where nothing is rendered (what would be the equator of the sphere). This is the case as the vertices of the rendered vertex are each in a different hemisphere and as each hemisphere is rendered separately the vertexes with the missing vertices must be discarded. On the other hand, Gnomonic projecting does ensure that the interpolation is non-linear, ensuring the depth of vertices is correct in proportion to others.

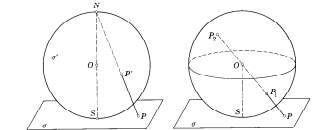


Figure 5: Stereographic (left) and Gnomonic (Right) Projection (Coxeter, 1989).

In order to project a Spherical non-Euclidean geometry full in one render pass, Stereographic projection can be used. This form of projection is very similar to that of Gnomonic projection. However, with Stereographic projection the origin point is not the centre of the sphere, instead, it is the point furthest away from the plane. This allows for the mapping of all points, except the origin onto a flat plan from the spherical geometry (Coxeter, 1989), see *figure 4 Stereographic Projection*. Utilising this form of projection, however, can cause similar depth issues as within Poincare Projection, this on the other hand, will only begin to become apparent in the hemisphere closest to the origin.

### Examples of Non-Euclidean Games

Hyperbolica (CodeParade, 2022) is a Game that utilises non-Euclidean geometry in both Spherical and Hyperbolic forms. This is an example of how games can work in a non-Euclidean format. Showing, that whilst the standards of the industry are to present games in a Euclidean geometry, it is not only possible to make a game in non-Euclidean geometry but possible to have positive feedback from the community surrounding the game. However, due to Hyperbolica’s small development team, it is obvious in the game has a very small scope. Despite this, the game is a very high-quality example of how various forms of non-Euclidean geometry can be used. Showing gameplay in various situations such as a shooting, flying and quests, proving that non-Euclidean geometry in practical in various situations.

A highly successful example of a non-Euclidean game series is the Portal Game series. The Portal games offer the non-Euclidean experience through the use of portals. These can be placed on surfaces and when passed through move the player to a different part of the map (where they placed the other portal). This means that whilst the world is Euclid (no curvature) the use of portals means the game cannot be interpreted in a world such as ours due to the inconsistency caused. This is due to the fact that in games such as Portal (Valve, 2007) or Portal 2 (Valve, 2011), the use of a portal can make distances and angles inconsistent, for example, if a point is 20 metres away from the player and they place a portal at that point, as well as one at 1 metre away from them, then they are both 20 metres and 1 metre away from the point. This is where the non-Euclidean aspect of the games appears. With these inconsistencies it is believable that the non-Euclidean nature of the could have caused audiences to find the game confusing, however, the Portal (Valve, 2007), holds extremely good reviews. Metacritic has awarded the game a 90 / 100 based on 27 Critic reviews, as well as a 9.2 / 10 based on 5930 user reviews. This demonstrates that games using a non-Euclidean format have the possibility to succeed in the games industry.

Another game that Utilises non-Euclidean geometry is the simple web game Hyperbolic Maze. This game shows how the use of different projections can change the resulting appearance of the game when using a Hyperbolic Geometry. In this game, the user can explore a 2D, hyperbolic maze, and choose between two different projections, Poincaré Disk and Beltrami-Klein, see *figure 5.* Each projection type represents the space differently on the screen. This demonstrates the importance of using the correct form of projection for the scenario, with some forms of projection being better for Spherical non-Euclidean geometry and others being better for Hyperbolic non-Euclidean geometry.

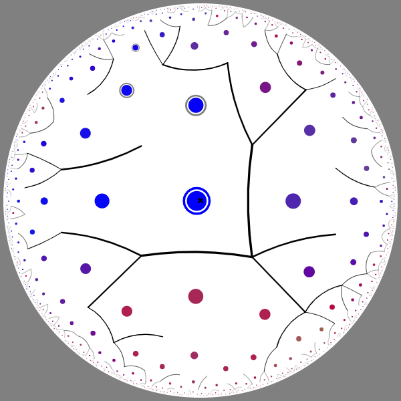
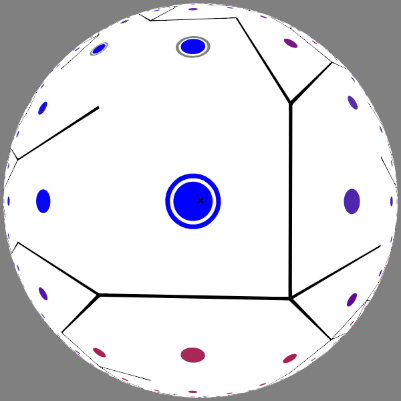


Figure 6: Poincaré Disk (left) and Beltrami-Klein(right) projections of the Hyperbolic Maze (Madore, 2013).



Figure 7: Octahedral holonomy maze (Segerman, N.D).

Not only aspects of non-Euclidean geometry are represented in virtual environments but also in the real world, *figure 6* shows a 3D printed maze that demonstrates the concept of holonomy. With this the green piece (rook) needs to be removed from the sphere, this can only be done by having it at the start point but with a different rotation. To achieve this the user put move the rook around the maze; due to the concept of holonomy, the rook will rotate based on its movement. With the addition of the blocking pieces on the maze, the user must move the rook in a specific way in order to achieve the required end rotation. This shows how non-Euclidean (curved) geometry causes rotation to be accumulated as things move along its surface, with the Octahedral shown in *figure 6* being equivalent to a positively curved 2D plane.

## Summary

One of the key parts of this investigation is to analyse the uses of non-Euclidean geometry, namely Hyperbolic and Spherical geometry, regarding the games industry. In this, the investigation has found many uses not only in the games industry but also for simulating real-world environments. This is becoming further apparent if we consider the applications of Space-Time curvature (Fiscaletti and Soril, 2014) as the bending of Space-Time can be considered as the warping of a flat plane.

One of the most common uses of Hyperbolic geometry is in the creation of tile-based games that have a greater number of tiles than that a Euclidean plane could provide. This means that the interesting nature of the geometry can help to counter some of the less interesting natures of games. “That would not be very interesting in a Euclidean tiling, but hyperbolic geometry makes it fun—not only because it feels very different, but also because it is possible to use the hyperbolic properties tactically, to deal with large groups of enemies” (Kopczynski et al, 2017). This means that Hyperbolic geometry can be used to make simple games far more interesting, in the case of HyperRogue (Kopczynski et al, 2017) this is done by having simple AI that moves towards the player, as a form of attack, but increases the number of enemies to account for their lack of depth. Due to the hyperbolic geometry, however, the player can still keep up with the hordes of enemies due to the increased range of movement the increased tile count of Hyperbolic geometry provided.

In regard to Spherical geometry, the primary and most obvious uses lie within simulating things that lie on a curved surface, such as the earth. With this, it becomes possible to map the curved nature of our planet and then using projections render this environment on a 2D screen. Whilst not having large scale effects on the games industry as a whole this already has an existing presence within the world of navigation. As a result of Euclid’s definition “A line which lies evenly between its extreme points is a straight or right line” (Casey, 2007). Whilst navigating on large distances the effect of the Earth’s curvature must be taken into account, as viewing the globe in a Euclid 3D space, we would assume the most direct path to be a flat straight line when in fact it is a spherical straight line.

However, the application to games begins to show itself when taking into account the curvature a Spherical geometry provided. This is primarily due to the unique mechanics that could be created from seeing further away geometry as larger (flipped if after the equator). This use of spherical geometry is shown in the game Hyperbolica (CodeParade, 2020). With this form of geometry, it is practical to say that strategy games could succeed extremely well if this is utilised correctly.

# Output Design

## Overview

Using the information gathered in the literature review of this paper a game will be created that utilises spherical non-Euclidean geometry. This geometry will contain a simple randomly generated maze that will allow the user to experience the spherical non-Euclidean Geometry in an interesting way.

## Possible Issues

### Level Creation

As a result of the Spherical Non-Euclidean nature of the game, the issue will arise regarding the creation of levels may arise. At this current point, the idea behind the game is that the levels will be procedurally generated, to ensure the mazes are different every time the game is played. However, due to the positive curvature found within Spherical Non-Euclidean Geometry simply designing the game in Euclidean Space and then rendering it Spherical Space issues may arise. If this is the case, then it is likely that a tile map-like system will have to be created.

If a tile made is to be used, the common method of simply using x, y and z coordinates is not practical. This is because the spherical plane in geometry will have to have a different number of, in this case, squares on each vertex to account for its curvature, three for spherical when flat would have four. This will in turn result in a custom coordinate system being created that will simply start at the origin and use up, down left and right to determine specific points on the geometry. If this is the case then it is likely that simple vectors will not be practical to use, due to the holonomy that this form of geometry causes. If this is the case, a simple fix can be created by introducing gyrovectors to the game, which will fix these issues, primary since their translations are non-commutative.

### Rendering

As a result of the Spherical Non-Euclidean geometry in the game, the method in which the game is projected will have to be changed. Since the game will use Spherical geometry, it is likely that Geometric projection will be used. The issue with this form of projection lies in rendering both hemispheres, this is because one render pass can only render one half of the geometry. If two render passes were used, one for each half of the geometry, the nature of Geometric projection will cause the line where these two render passes meet to not render, as both passes can only render up to one half of the geometry. If two vertices of each triangle are in different hemispheres to each other then the whole vertex will be discarded, resulting in nothing being drawn.

As a result of the issues mentioned Geometric projection cannot be used in this case. This resulted in the conclusion that Stereographic projection must be used as it renders the whole geometry in one pass. However, it is likely that depth issues will arise if this method is used, with these issues becoming worse the further the geometry is from the camera. To fix this, two render passes can be used, both stereographic, one rendering the closer geometry and one rendering the further.

There is still one issue that may arise when rendering the geometry for the game. This is the case due to the stereographic projection that will be used. If this form of projection does have to be utilised then the depth buffer will have to be altered to compensate for issues in the far hemisphere, as well as the joining of the two projections. To do this the two projections will be in one depth buffer, the near occupying 0 – 0.5 (mapped 0 to 0 and 1 to 0.5), whilst the far occupies 0.5 to 1 (mapped 0 to 1 and 1 to 0.5 to account for depth issues).

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